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COMMENT

Electrostatics of two unequal adhering spheres

D J Jeffrey and Y Onishi†

Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge

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Abstract. The method recently put forward for finding the electrostatic fields around two touching spheres must be generalised to include the possibility that a small residual gap remains between the spheres. The reason is that the calculated values of some quantities are significantly affected if such a gap exists and, since a real physical system can only approximate mathematical spheres in point contact, it is important to be able to calculate deviations from ideal behaviour. The generalisation is illustrated using four different problems.

1. Introduction

Moussiaux and Ronveaux (1979) have given a formal expression for the electrostatic field around two unequal spheres in point contact, and used it to calculate the capacitance of two unequal adhering spheres. They stated that their general method could be applied both to other charge configurations and to the calculation of other quantities. Although their statement is formally correct, it is important to make the proviso that some calculations are very sensitive to slight deviations from the assumed geometry. We illustrate this by solving three new electrostatic problems, calculating each time the spheres' charges and dipole moments, and showing that some quantities depend in a singular way on the size of any small residual gap between the spheres. We also give new simple expressions for the charge and dipole moment of a sphere.

We define four electrostatic problems: (a) the spheres are at the same potential; (b) the spheres are at opposite potentials; (c) grounded spheres are in an applied field parallel to the line of centres and (d) grounded spheres are in an applied field perpendicular to the line of centres. Problem (b) is strictly only defined when there is a gap between the spheres, but in spite of this the general method for touching spheres gives a formal solution for the field, which we later supplement with a separate analysis of the gap region. Problem (a) was solved by Moussiaux and Ronveaux (1979), although they did not calculate the dipole moments of the spheres. Because electrostatic fields can be superimposed, the singularity in problem (b) can appear in other problems, for example, a problem (bc) in which the charge on each sphere is given and their potentials are determined by an applied field (and the size of the small gap).

† Permanent address: Department of Mechanical Engineering, University of Osaka Prefecture, Sakai, Japan.

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2. The expressions for the fields

Let the spheres have radii a and b, let $\lambda = a/b$ and let $\omega = b/(a+b)$. The point at which the spheres touch is taken as the origin for a set of cylindrical polar coordinates (ar, az, θ) with the z axis along the line of centres. Tangent-sphere coordinates (ξ, η, θ) are defined by

$$z = 2\xi/(\xi^2+\eta^2), \qquad r = 2\eta/(\xi^2+\eta^2).$$

This coordinate system has been amply illustrated in Smith and Barakat (1975), Moussiaux and Ronveaux (1979) and Moon and Spencer (1961) and will not be described further here. The general solution of Laplace's equation appropriate to problems (a)–(c) is

$$(\xi^2+\eta^2)^{1/2}\int_0^\infty \{f(s,\lambda)\sinh s\xi + g(s,\lambda)\cosh s\xi\} J_0(s\eta)\,\mathrm{d}s. \tag{1}$$

For problems (a) and (b) we set the potential equal to $+1$ on the top sphere (radius a) and ± 1 on the bottom sphere, and obtain the following expressions for f and g, the upper sign applying to problem (a).

$$f\sinh(1+\lambda)s = \mathrm{e}^{-s}\cosh\lambda s \mp \mathrm{e}^{-\lambda s}\cosh s,$$

$$g\sinh(1+\lambda)s = \mathrm{e}^{-s}\sinh\lambda s \pm \mathrm{e}^{-\lambda s}\sinh s.$$

For problem (c) the potential at infinity obeys $\phi_c \sim -E_c az$ and our general solution (1) now applies to $\phi_c + E_c az$, with f_c and g_c given by

$$f_c\sinh(1+\lambda)s = 2saE_c(\mathrm{e}^{-s}\cosh\lambda s + \mathrm{e}^{-\lambda s}\cosh s),$$

$$g_c\sinh(1+\lambda)s = 2saE_c(\mathrm{e}^{-s}\sinh\lambda s - \mathrm{e}^{-\lambda s}\sinh s).$$

Finally, for problem (d), we have

$$\phi_d = aE_d\cos\theta\left[-r+(\xi^2+\eta^2)^{1/2}\int_0^\infty \{f_d\sinh s\xi + g_d\cosh s\xi\}J_1(s\eta)\,\mathrm{d}s\right],$$

with f_d and g_d given by

$$f_d\sinh(1+\lambda)s = 2s(\mathrm{e}^{-s}\cosh\lambda s - \mathrm{e}^{-\lambda s}\cosh s),$$

$$g_d\sinh(1+\lambda)s = 2s(\mathrm{e}^{-s}\sinh\lambda s + \mathrm{e}^{-\lambda s}\sinh s).$$

For the special case $\lambda = 1$, the above expressions agree with corresponding ones given in Jeffrey (1978), when we note that the directions of the fields in (c) and (d) have been reversed.

3. Charges on the spheres

In SI units the charge on the upper sphere is

$$Q^u = a\epsilon_0\int_0^{2\pi}\int_0^\infty [\partial\phi/\partial\xi]_{\xi=1}[2\eta\,\mathrm{d}\eta\,\mathrm{d}\theta/(1+\eta^2)], \tag{2}$$

while that on the lower sphere is

$$Q^{\mathrm{l}} = -a\epsilon_0 \int_0^{2\pi} \int_0^{\infty} [\partial\phi/\partial\xi]_{\xi=-\lambda}[2\eta \; \mathrm{d}\eta \; \mathrm{d}\theta/(\lambda^2+\eta^2)]. \tag{3}$$

Substituting the general expression (1) into (2) and (3), we find that the charges in cases (a)–(c) are given by the simple formulae

$$Q = 4\pi\epsilon_0 a \int_0^{\infty} (g \pm f) \; \mathrm{d}s,$$

where the + sign is taken for the upper sphere. Now substituting the explicit expressions for f and g, we arrive at the following results. For problem (a), the charge on the upper sphere is

$$Q_{\mathrm{a}}^{\mathrm{u}} = 4\pi\epsilon_0 a(1+\lambda)^{-1}[\psi(1) - \psi(\omega)],$$

where ψ is defined in Gradshteyn and Ryzhik (1965, § 8.361), and the charge on the lower sphere $Q_{\mathrm{a}}^{\mathrm{l}}$ is obtained by replacing $\psi(\omega)$ by $\psi(1-\omega)$. The sum $Q_{\mathrm{a}}^{\mathrm{u}} + Q_{\mathrm{a}}^{\mathrm{l}}$ agrees with existing results, when allowance is made for different units. Problem (b) gives an integrand that is not integrable and so is treated separately. For problem (c),

$$Q_{\mathrm{c}}^{\mathrm{u}} = 4\pi\epsilon_0 a^2(1+\lambda)^{-2}[\tfrac{1}{6}\pi^2 + \zeta(2, \omega)]E_{\mathrm{c}},$$

and $-Q_{\mathrm{c}}^{\mathrm{l}}$ is obtained by replacing $\zeta(2, \omega)$ by $\zeta(2, 1-\omega)$, the function ζ being defined in Gradshteyn and Ryzhik (1965, § 9.521). For problem (d) the charge integrates to zero.

4. The singular problem (b)

If the spheres are separated by a small gap of width $2a\delta$, with $\delta \ll 1$, we can find expressions for Q_{b} which are singular as $\ln \delta$. Outside the gap, the expression (1) is an approximation correct to $\mathrm{O}(\delta)$, provided $\eta < \eta_0$, where η_0 is a measure of the 'edge' of the gap. Inside the gap we define stretched coordinates by (Jeffrey 1978)

$$Z = z/\delta, \qquad R = r/\delta^{1/2}.$$

The upper surface of the gap is then given to $\mathrm{O}(\delta)$ by $Z = 1 + \frac{1}{2}R^2$, the lower by $Z = -1 - \frac{1}{2}\lambda R^2$. The potential in the gap satisfies $\partial^2\Phi/\partial Z^2 = 0$ and is

$$\Phi = [1 + \tfrac{1}{4}(1+\lambda)R^2]^{-1}[Z + \tfrac{1}{4}(\lambda - 1)R^2] + \mathrm{O}(\delta). \tag{4}$$

This expression is valid for $R < R_0$, where R_0 is another measure of the edge of the gap. The charge on the upper surface of the gap is

$$Q_{\mathrm{b}}^{\mathrm{u}}(\mathrm{gap}) = 2\pi\epsilon_0 a \int_0^{R_0} (\partial\Phi/\partial Z)R \; \mathrm{d}R = 4\pi\epsilon_0 a(1+\lambda)^{-1} \ln[1 + \tfrac{1}{4}(1+\lambda)R_0^2].$$

We note that the contribution is singular as $\ln R_0$. We now cancel this singularity with a similar one in the contribution to $Q_{\mathrm{b}}^{\mathrm{u}}$ from outside the gap. On the top sphere we write:

$$[\partial\phi/\partial\xi]_{\xi=1} = [2/(1+\lambda)] + (1+\eta^2)^{-1/2} \int_0^{\infty} [f_{\mathrm{b}} \sinh s + g_{\mathrm{b}} \cosh s]J_0(s\eta) \; \mathrm{d}s$$

$$+ (1+\eta^2)^{1/2} \int_0^{\infty} [sf_{\mathrm{b}} \cosh s - 2(1+\lambda)^{-1} \, \mathrm{e}^{-s} + sg_{\mathrm{b}} \sinh s]J_0(s\eta) \; \mathrm{d}s.$$

When we substitute this into (2), we find that the first term leads to a singularity, and so we integrate it only to η_0, the edge of the gap. The remaining terms we integrate as before to obtain

$$Q_b^u(\text{sphere}) = 4\pi\epsilon_0 a\left\{(1+\lambda)^{-1}\ln(1+\eta_0^2) + \int_0^\infty [f_b + g_b - 2(1+\lambda)^{-1}s^{-1}\,e^{-2s}]\,ds\right\}.$$

Using the relation $\delta^{1/2}R_0 = 2\eta_0/(1+\eta_0^2)$ to add the two contributions together, we obtain, on performing the integrations,

$$Q_b^u = 4\pi\epsilon_0 a(1+\lambda)^{-1}[\ln\delta^{-1} - \ln(1+\lambda) - \psi(1) - \psi(\omega)] + O(\delta).$$

The singular dependence on the gap width is thus clearly seen. A parallel set of manipulations for the lower sphere shows that $-Q_b^l$ can be obtained by replacing $\psi(\omega)$ by $\psi(1-\omega)$.

5. Dipole moment

The final quantity we calculate is the dipole moment, defined by

$$\boldsymbol{S} = -\epsilon_0\int \boldsymbol{x}\,\partial\phi/\partial n\,dS,$$

where $\partial/\partial n$ is the outward normal derivative. For problems (a)–(c) only the z component of $\boldsymbol{S}$ is non-zero and for the upper sphere it is

$$S^u = 2\pi\epsilon_0 a^2\int_0^\infty [\partial\phi/\partial\xi]_{\xi=1}[4\eta\,d\eta/(1+\eta^2)^2].$$

Substituting the general expression (1), we obtain

$$S^u = \tfrac{8}{3}\pi\epsilon_0 a^2\int_0^\infty [sf + f\,e^{-s}(\sinh s + 2s\cosh s) + sg + g\,e^{-s}(2s\sinh s + \cosh s)]\,ds$$
$$+\{\tfrac{4}{3}\pi\epsilon_0 a^3 E_c\},$$

where the final term is needed for problem (c) and is the contribution from the field $-aE_c z$. Similarly, we obtain

$$S^l = \tfrac{8}{3}\pi\epsilon_0 a^2\int_0^\infty [sf + f\,e^{-\lambda s}(\lambda^{-1}\sinh\lambda s + 2s\cosh\lambda s) - sg$$
$$-g\,e^{-\lambda s}(2s\sinh\lambda s + \lambda^{-1}\cosh\lambda s)]\,ds + \{\tfrac{4}{3}\pi\epsilon_0\lambda^{-3}a^3E_c\}.$$

Finally, using the explicit forms for f and g, we arrive at

$$S_a^u = 4\pi\epsilon_0 a^2(1+\lambda)^{-2}[\zeta(2,\omega) - \tfrac{1}{6}\pi^2],$$
$$S_b^u = 4\pi\epsilon_0 a^2(1+\lambda)^{-2}[\zeta(2,\omega) + \tfrac{1}{6}\pi^2],$$
$$S_c^u = 8\pi\epsilon_0 a^3(1+\lambda)^{-3}[\zeta(3,\omega) + \zeta(3)]E_c.$$

To obtain S^l from these formulae, change ω to $1-\omega$ and change the sign of S_a.

For problem (d), $\boldsymbol{S}$ is parallel to $\boldsymbol{E}_{\mathrm{d}}$ and has magnitude

$$S_{\mathrm{d}} = \epsilon_0 a^2 \int_0^{2\pi} \int_0^{\infty} [\partial\phi_{\mathrm{d}}/\partial\xi]_{\xi=1}(4\eta^2 \cos\theta/(1+\eta^2)^2)\, \mathrm{d}\eta\, \mathrm{d}\theta$$

for the upper sphere. This leads to the explicit formula

$$S_{\mathrm{d}}^{\mathrm{u}} = 4\pi\epsilon_0 a^3(1+\lambda)^{-3}[\zeta(3,\omega)-\zeta(3)]E_{\mathrm{d}}.$$

We can obtain $S_{\mathrm{d}}^{\mathrm{l}}$ by changing ω to $1-\omega$.

6. The effects of deviations from point contact

Let us define a problem in which spheres with fixed zero charge are placed in the applied field of problem (c), and we ask for the potential V_{bc} of each sphere. We write $V_{\mathrm{bc}}^{\mathrm{u}} Q_{\mathrm{b}}^{\mathrm{u}} + Q_{\mathrm{c}}^{\mathrm{u}} = 0$ and solve for $V_{\mathrm{bc}}^{\mathrm{u}}$.

$$V_{\mathrm{bc}}^{\mathrm{u}} = -aE_{\mathrm{c}}(1+\lambda)^{-1}[\tfrac{1}{6}\pi^2+\zeta(2,\omega)]/[\ln\delta^{-1}-\ln(1+\lambda)-\psi(1)-\psi(\omega)].$$

We note that $\lim_{\delta\to 0} V_{\mathrm{bc}}^{\mathrm{u}} = 0$, but the dependence on $\ln\delta^{-1}$ means that even very small gaps will produce a significant deviation from zero. For example if $\delta = 0{\cdot}001$ and $\lambda = 1$, then $V_{\mathrm{bc}}^{\mathrm{u}} = -0{\cdot}38aE_{\mathrm{c}}$. A similar comment applies to the dipole moment S_{bc}, given by

$$S_{\mathrm{bc}} = V_{\mathrm{bc}} S_{\mathrm{b}} + S_{\mathrm{c}}.$$

We note that it was not necessary to have a gap to keep S_{b} finite, but even so S_{bc} remains sensitive to its existence, for example, if $\lambda = 1$ then $S_{\mathrm{bc}}(\delta=0) = 9{\cdot}6\pi a^3\epsilon_0 E_{\mathrm{c}}$, but $S_{\mathrm{bc}}(\delta = 0{\cdot}001) = 7{\cdot}1\pi a^3\epsilon_0 E_{\mathrm{c}}$.

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